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# Concerning the Constants in Fuzzy Topology

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## 1. INTRODUCTION

Since the introduction of the first paper combining fuzzy sets with topological ideas [1] many papers have appeared using the definition of "fuzzy topology" given in that very first paper. The notion defined there we refer to as a quasi-fuzzy topology. Later [14] a more restrictive concept was introduced which we call a fuzzy topology.

In the presence of a rather overwhelming evidence in favor of this latter notion, demonstrated in [2, 5–8, 11–33, 37–41], nevertheless many papers keep appearing which explicitly use the notion of quasi-fuzzy topology. None of these papers however has presented an argument why quasi-fuzzy topologies, which are not fuzzy topologies, are interesting.

Apart from the goal of listing some new, and also some of the most important and known advantages of fuzzy topologies over quasi-fuzzy topologies, bundled in one paper, our secondary purpose with this work is precisely to prompt the question of whether there exist mathematically important reasons for a continued study of quasi-fuzzy topologies.

The fact that we restrict ourselves, as usual, to [0, 1]-valued fuzzy sets, has no consequence on the fundamental aspects of this paper. The ideas can and should be interpreted in their largest possible context.

## 2. PRELIMINARIES

As usual I denotes the unit interval. If $X$ is a set then a quasi-fuzzy topology on $X$—as introduced by Chang in [1]—is simply a collection of fuzzy sets on $X$, stable for arbitrary suprema and finite infima and containing the constant fuzzy sets 0 and 1. A fuzzy topology [14] on $X$ is a quasi-fuzzy topology which moreover contains all constant fuzzy sets. A probabilistic (or translation closed fuzzy) topology is a fuzzy topology which together with any open fuzzy set $\mu$ also includes $(\mu+\alpha) \wedge 1$ and $(\mu-\alpha) \vee 0$ for any constant fuzzy set $\alpha$. This notion was introduced by Höhle in [6].



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Q-FTS, FTS, and TOP denote the categories of, respectively, quasi-fuzzy topological, fuzzy topological, and topological spaces. The following functors shall be used:

$$\omega: \mathrm{TOP} \to \mathrm{FTS}$$

$$(X, \mathscr{T}) \to (X, \omega(\mathscr{T})),$$

where $\omega(\mathscr{T}) := \{\mu \in I^X \mid \mu \text{ lsc}\}$,

$$\lambda: \mathrm{TOP} \to \text{Q-FTS}$$

$$(X, \mathscr{T}) \to (X, \lambda(\mathscr{T})),$$

where $\lambda(\mathscr{T}) := \{1_G \mid G \in \mathscr{T}\}$,

$$\iota: \text{Q-FTS} \to \mathrm{TOP}$$

$$(X, \Delta) \to (X, \iota(\Delta)),$$

where $\iota(\Delta) :=$ coarsest topology on $X$ making all fuzzy sets in $\Delta$ lsc, and where all three functors leave morphisms unchanged. The restriction of $\iota$ to FTS is also denoted simply $\iota$. Remark that $\iota$ is a left inverse for both $\omega$ and $\lambda$.

A fuzzy topological group is a group equipped with a fuzzy topology making the group operations continuous. Analogously a fuzzy topological vector space is a vector space equipped with a fuzzy topology making addition and scalar multiplication continuous. These notions were introduced respectively by Foster in [2] and Katsaras in [11].

The following concepts were introduced by Herrlich in [3]. A concrete category is a category whose objects are structured sets, i.e., pairs $(X, \Delta)$, where $X$ is a set and $\Delta$ is a structure on $X$, whose morphisms $f$: $(X, \Delta) \to (Y, \Gamma)$ are certain functions between $X$ and $Y$, and whose composition law is the usual composition of functions. A concrete category is called topological if it satisfies the following conditions:

(1) (Existence of initial structures). For any set $X$, any family $(X_j, \Delta_j)_{j \in J}$ of objects, and any family $(f_j: X \to X_j)_{j \in J}$ of functions there exists a unique structure $\Delta$ on $X$ which is initial with respect to $(X, f_j, (X_j, \Delta_j))_{j \in J}$, i.e., such that for any object $(Y, \Gamma)$ a map $g: (Y, \Gamma) \to (X, \Delta)$ is a morphism iff for every $j \in J$ the composite map $f_j \circ g: (Y, \Gamma) \to (X_j, \Delta_j)$ is a morphism.

(2) (Fibre smallness). For any set $X$, the fibre of $X$, i.e., the class of all structures on $X$, is a set.

(3) (Terminal separator property). For any set $X$ with cardinality one, there exists precisely one structure on $X$.

## 3. The Arguments

**3.1.** *FTS is, whereas Q-FTS is not, a topological category.*

*Proof.* That both FTS and Q-FTS are concrete categories is clear. Further, in both categories initial structures exist. For FTS this was shown in [15] and, since there the constants play no part in proofs, it follows that the result also holds in Q-FTS. Also both categories are evidently fibre small. However, only FTS fulfils the terminal separator property. If $X$ is a one-point set then the only fuzzy topology on $X$ is $I^X \approx I$, whereas uncountably many quasi-fuzzy topologies exist on $X$.

**3.2.** *Constant functions are morphisms in FTS but not necessarily in Q-FTS.*

*Proof.* That constant functions are morphisms in FTS is an immediate consequence of the fact that FTS is a topological category [3]. To show that this is not necessarily so in Q-FTS simply take the following counterexample.

Let $X \in$ Q-FTS\FTS and let $Y \in$ FTS, then if $\alpha$ is a constant, not open in $X$, and $f$ is any map $X \to Y$ it follows from $f^{-1}(\alpha) = \alpha$ that $f$ is not continuous. So not only are constant maps $X \to Y$ not continuous but actually $\mathrm{Hom}(X, Y) = \emptyset$! This at the same time shows that Q-FTS is not a connected category and that FTS is a maximal subcategory of Q-FTS in which this pathology can be avoided.

**3.3.** *Any nonempty object in FTS is a separator but no object in Q-FTS\FTS is a separator.*

*Proof.* For FTS this follows again from [3]. For Q-FTS\FTS take $X \in$ Q-FTS\FTS, $X \neq \emptyset$, and $Y \in$ FTS, then $\mathrm{Hom}(X, Y) = \emptyset$, and thus $X$ cannot be a separator.

**3.4.** *For product spaces projections are open in FTS but not necessarily in Q-FTS.*

*Proof.* Let $(X_j)_{j \in J}$ be a family of fuzzy topological spaces ($X_i$ not a singleton), $p_i\colon \prod_{j \in J} X_j \to X_i$ the $i$th projection, and $\inf_{k=1}^{n} p_{j_k}^{-1}(\mu_{j_k})$ a basic open fuzzy set in $\prod_{j \in J} X_j$. Then using the fact that if $\xi \neq \eta \in X_i$ then the map $\theta\colon p_i^{-1}(\xi) \to p_i^{-1}(\eta)$ defined by

$$\theta(x)_j = \begin{cases} \eta & j = i, \\ x_j & j \neq i, \end{cases}$$

clearly is a bijection, the reader can easily verify that

$$p_i\left(\inf_{k=1}^{n} p_{j_k}^{-1}(\mu_{j_k})\right)(\xi)=p_i\left(\inf_{k=1}^{n} p_{j_k}^{-1}(\mu_{j_k})\right)(\eta)$$

in case $i\notin\{j_1,\dots,j_n\}$ and

$$\begin{aligned} p_i\left(\inf_{k=1}^{n} p_{j_k}^{-1}(\mu_{j_k})\right)(\xi)&=\mu_i(\xi)\wedge\left(p_i\left(\inf_{\substack{k=1\\k\neq m}}^{n} p_{j_k}^{-1}(\mu_{j_k})\right)(\xi)\right)\\ &=\mu_i(\xi)\wedge\left(p_i\left(\inf_{\substack{k=1\\k\neq m}}^{n} p_{j_k}^{-1}(\mu_{j_k})\right)(\eta)\right)\end{aligned}$$

in case $i=j_m$, for some $m$. This simply means that there are constants $\alpha$ and $\beta$ such that

$$p_i\left(\inf_{k=1}^{n} p_{j_k}^{-1}(\mu_{j_k})\right)=\begin{cases}\alpha & \text{if } i\notin\{j_1,\dots,j_n\}\\ \mu_i\wedge\beta & \text{if } i\in\{j_1,\dots,j_n\}\end{cases}.$$

Consequently it follows that in general projections can be open only in FTS.

An explicit counterexample in Q-FTS is given by taking $X_1$ and $X_2$ arbitrary and respectively endowed with the quasi-fuzzy topologies $\{0,\frac{1}{2},1\}$ and $\{0,1\}$ . Then $p_2$ is not open.

**3.5.** *In product spaces slices are homeomorphic to factors in FTS but not necessarily in Q-FTS.*

What this precisely means is that if $j_0\in J$ and $x^0\in\prod_{j\in J}X_j$ are fixed and we put

$$S(x^0;j_0)=X_{j_0}\times\prod_{j\neq j_0}\{x_j^0\}$$

then the map: $X\to S(x^0;j_0)$: $\xi\to\varphi(\xi)$ defined by

$$\varphi(\xi)_j=\begin{cases}\xi & j=j_0\\ x_j^0 & j\neq j_0\end{cases}$$

is a homeomorphism when $S(x^0;j_0)$ is endowed with the product topology. The proof is very similar to that of 3.4 and will be left to the reader.

That this property does not hold in Q-FTS is shown using the same counterexample as in 3.4. It is then easily seen that no slice parallel to $X_2$, i.e., $S(x^0;2)$ for some $x^0\in X_1\times X_2$, is homeomorphic to $X_2$. The positive results of 3.4 and 3.5 in FTS were also noted by Pu and Liu in [31, 32].

**3.6.** *Many results relating properties of the product space to the same properties of the factor spaces and which hold in TOP carry over to FTS but fail in Q-FTS.*

We do not intend to give an account of all such results here. They are scattered over the literature. However, simply note that many proofs of such properties only use the openness of the projections and/or the fact that slices are homeomorphic to factors. If so they carry over to FTS in a trivial way. Counterexamples that this is not the case in Q-FTS usually are easy to construct, as before, or else can also be found here and there in the literature.

**3.7.** *Fuzzy structures compatible with a vector space or group structure are translation invariant only in FTS.*

These assertments were very clearly shown for groups by Foster in [2] and for vector spaces by Katsaras in [11]. We refer the reader to these papers.

**3.8.** *Convergence is pathological in Q-FTS.*

*Proof.* This assertment is true both for the convergence theory developed in [16] and for the convergence theory developed in [32, 33]. Although the former theory was conceived and used only in FTS its definition can be extended to Q-FTS without altering one single concept, and it is that which we shall do here without further mention. We shall refer to this theory as the filter convergence and to the theory of [32, 33] as the net convergence.

To prove our claim some weird examples suffice. Let $(X, \mathscr{T})$ be any topological space and then consider the quasi-fuzzy topological space $(X, \lambda(\mathscr{T}))$ . Now consider the sequence

$$\left(\frac{1}{n} 1_{x_n}\right)_{n \in \mathbb{N}_0}, \qquad \text{where} \quad x_n \to x \ \text{in} \ (X, \mathscr{T})$$

and denote $\mathfrak{F}$ the prefilter generated by this sequence, i.e.,

$$\mathfrak{F} := \left[\left\{\sup_{k \geqslant n} \frac{1}{k} 1_{x_k} \mid n \in \mathbb{N}_0\right\}\right].$$

Applying filter convergence in $(X, \lambda(\mathscr{T}))$ to $\mathfrak{F}$ we find

$$\lim \mathfrak{F} = 1_x$$

and applying net convergence in $(X, \lambda(\mathscr{T}))$ to $((1/n)1_{x_n})_{n \in \mathbb{N}_0}$ we find

$$\frac{1}{n} 1_{x_n} \to \alpha 1_x \qquad \text{for any} \quad \alpha \in I_0.$$

This is indeed pathological since $((1/n)1_{x_n})_{n\in\mathbb{N}_0}$ is really a sequence of "vanishing points." Think of a snowflake falling down while slowly disintegrating; or a waterdrop on a hot tilted plate. Both these phenomena might be described by such a sequence. However, the limit of the former is not a snowflake and of the latter is not a waterdrop. If however we look at the same sequence and associated prefilter in $(X, \omega(\mathscr{T}))$ then we find for net convergence that

$$\frac{1}{n}1_{x_n} \qquad \text{converges to no fuzzy point } \alpha 1_x$$

and for filter convergence that

$$\lim \mathfrak{F} = 0.$$

Indeed, in the limit, the waterdrop, for example, will have disappeared completely.

The reason for the pathology in Q-FTS\FTS is that there we do not have natural bounds on the "height" of lim $\mathfrak{F}$ of adh $\mathfrak{F}$ depending on the given prefilter $\mathfrak{F}$ nor on the "height" of limit or adherence points of a given net and depending on this net.

For any prefilter $\mathfrak{F}$, in [16, 18] the characteristic and lower characteristic value of $\mathfrak{F}$ were defined respectively as

$$\begin{aligned} c(\mathfrak{F}) &:= \inf_{\mu\in\mathfrak{F}} \sup_{x\in X} \mu(x) \\ &= \inf\{\alpha \mid \alpha \text{ constant}, \alpha\in\mathfrak{F}\} \\ &= \sup_{\mathfrak{G}\in\mathscr{P}_m(\mathfrak{F})} c(\mathfrak{G}) \end{aligned}$$

and

$$c^-(\mathfrak{F}) := \inf_{\mathfrak{G}\in\mathscr{P}_m(\mathfrak{F})} c(\mathfrak{G}),$$

where $\mathscr{P}_m(\mathfrak{F})$ denotes the family of all minimal prime prefilters finer than $\mathfrak{F}$ (see [16]). For any fuzzy net we have counterparts of these values.

Let $\mathfrak{N}$ be the fuzzy net

$$\mathfrak{N} := \{\alpha_d 1_{x_d} \mid d\in D\}$$

then we define

$$c(\mathfrak{N}) := \limsup_{d\in D} \alpha_d$$

$$c^-(\mathfrak{N}) := \liminf_{d\in D} \alpha_d.$$

We remark that $c^{-}(\mathfrak{N})$ was already defined in [26]. Then we now have the following result.

*In a fuzzy topological space $(X, \Delta)$ and for any fuzzy net $\mathfrak{N}$ and prefilter $\mathfrak{F}$ we have*

$1^\circ$ $\mathfrak{N}$ *converges to* $\alpha 1_x \Rightarrow \alpha \leqslant c^{-}(\mathfrak{N})$

$2^\circ$ $\mathfrak{N}$ *adheres to* $\alpha 1_x \Rightarrow \alpha \leqslant c(\mathfrak{N})$

$3^\circ$ $\lim \mathfrak{F} \leqslant c^{-}(\mathfrak{F})$

$4^\circ$ $\operatorname{adh} \mathfrak{F} \leqslant c(\mathfrak{F})$.

The proof of $1^\circ$ is given in Proposition 4.12 [26] while $2^\circ$ follows at once from the fact that $\mathfrak{N}$ adheres to $\alpha 1_x$ if and only if there exists a subnet of $\mathfrak{N}$ which converges to $\alpha 1_x$ [32, Theorem 13.2]. By definition $4^\circ$ is a triviality, and $3^\circ$ follows from this: from the fact that $\lim \mathfrak{F} = \inf_{\mathfrak{G} \in \mathscr{P}_m(\mathfrak{F})} \operatorname{adh} \mathfrak{G}$ and from the third characterization of $c(\mathfrak{F})$.

To end this section we point out that these results are equivalent, in terms of convergence, to the principle put forward by Weiss in [39], which says that a point should not belong to the closure of a fuzzy set with a degree higher than the supremum of that set, while in Q-FTS it can happen that the closure of a constant set $\alpha$ ($\alpha$ arbitrarily small!) becomes a set of the type $\alpha \vee 1_x$.

**3.9.** *Topologically generated spaces, probabilistic topological spaces, fuzzy neighborhood spaces, probabilistically uniformizable spaces, fuzzy uniform spaces and probabilistically metrizable spaces all automatically contain the constants and thus are in FTS.*

All these results are well known, and can be found in [6–8, 14, 17, and 18]. The fact that all these types of spaces by their defining mechanism automatically contain the constants is worthy of notice. Indeed in ordinary topology one often has to specify open sets in some canonical way and then, almost as an afterthought, one has to add also $\varnothing$ and $X$ as open sets unless one supposes them automatically included through the lattice properties. However, if the topology is determined by a richer structure, such as a uniformity, then $\varnothing$ and $X$ also are "automatically" open. The same happens in FTS: in case a richer structure determines the fuzzy topology, often the constants are automatically included. Thus one might argue that "adding $\varnothing$ and $X$" in topology generalizes to "adding the constants" in fuzzy topology. We remark moreover that both in TOP and FTS these are precisely the conditions which make the terminal separator property hold and which consequently turn these categories into topological categories.

**3.10.** *The "fuzzy real line" is generated by a canonical fuzzy order only in FTS and not in Q-FTS.*

In [34–36], Rodabaugh virtually claims that the "fuzzy real line" is a fundamental example of a quasi-fuzzy topological, nonfuzzy topological, space. In our opinion this is not quite so. The defining notions of the fuzzy real line are its underlying set and, originally, its defining subbase for the open fuzzy sets [9]. This subbase however can be saturated in Q-FTS [9], in FTS [24], in the category of probabilistic topological spaces [5], or even in the category of all topologically generated spaces (i.e., essentially in TOP!) [24]. The fact that it was first done in Q-FTS may be of some historical importance but seems to have no fundamental justification. This means that until now no reasons have been stated why the "fuzzy real line" is essentially more natural when considered in Q-FTS as to when considered in FTS. More importantly however there are very fundamental reasons which point in the opposite direction. The defining subbase of the "fuzzy real line" is determined by the order of $\mathbb{R}$ and not by the topology of $\mathbb{R}$, and only in FTS is it possible to extend the order of $\mathbb{R}$ in a canonical way to a fuzzy order on the underlying subset of the "fuzzy real line" and which in turn then again in a canonical way generates the fuzzy topology of the "fuzzy real line." The proof of all these claims can be found in [28]. Let us however recall the construction and main properties of this fuzzy order. We use here the more natural model of the underlying set of the "fuzzy real line" as introduced in [24], i.e., $\mathscr{M}(\mathbb{R})$, the family of all probability measures on $\mathbb{R}$.

For any $P$, $Q \in \mathscr{M}(\mathbb{R})$ let

$$\rho(P, Q) := \sup_{a \in \mathbb{R}} P(\,]-\infty, a[\,) \wedge Q(\,]a, +\infty[\,).$$

That this fuzzy relation can indeed be considered as an extension of the strict order relation on $\mathbb{R}$ is shown by the following properties it enjoys:

1° (Extension). For any pair of degenerate measures $P_x$ and $P_y$

$$\rho(P_x, P_y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x \geqslant y. \end{cases}$$

2° (Antisymmetry). For any $P, Q \in \mathscr{M}(\mathbb{R})$

$$T_m(\rho(P, Q), \rho(Q, P)) = 0.$$

3° (Transitivity). For any $P$, $Q \in \mathscr{M}(\mathbb{R})$

$$\rho(P, Q) \geqslant \sup_{R \in \mathscr{M}(\mathbb{R})} T_m(\rho(P, R), \rho(R, Q)).$$

4° (Linearity). For any $P \neq Q \in \mathcal{M}(\mathbb{R})$

$$S_m(\rho(P, Q), \rho(Q, P)) > 0.$$

Here, as usual, $T_m$ denotes the $t$-norm $T_m(a, b) = \max(a + b - 1, 0)$ and $S_m$ its dual $S_m(a, b) = \min(a + b, 1)$.

In FTS the fuzzy topology of the "fuzzy real line" is generated by the left and right sections of $\rho$ just as the topology of $\mathbb{R}$ is generated by the left and right sections of the strict order on $\mathbb{R}$. This also holds in the smaller category of probabilistic topological spaces. However, it fails in Q-FTS.

**3.11.** *For any topological space $(X, \mathcal{T})$, fuzzy closure is uniformly continuous in $\omega(\mathcal{T})$ but fails to be even pointwise continuous in $\lambda(\mathcal{T})$.*

The result for $\omega(\mathcal{T})$ follows at once from Theorem 5.3(i) [18], where the result was shown even for arbitrary fuzzy neighborhood spaces. Actually even more was shown, i.e., the mapping $\mu \to \bar{\mu}$ was shown to be Lipschitz for the uniform norms since for any $\mu, \nu \in I^X$ we had $\|\bar{\mu} - \bar{\nu}\| \leqslant \|\mu - \nu\|$.

However, consider the following example in Q-FTS. Let $X = \mathbb{R}$ and take again $\lambda(\mathcal{T}_{\text{usual}})$. Let $(\mu_n)_{n \in \mathbb{N}_0}$ be the sequence of fuzzy sets defined as

$$\mu_n := \frac{1}{n} 1_{\mathbb{Q}}$$

then clearly $\mu_n \to 0$ uniformly. Since however, in $\lambda(\mathcal{T}_{\text{usual}})$, $\bar{\mu}_n = 1$ for all $n \in \mathbb{N}_0$ and $\bar{0} = 0$ we see that $\bar{\mu}_n$ does not converge pointwise to 0 in any single point.

Another way of seeing this pathology in $\lambda(\mathcal{T})$ is as follows. Let $X$ be any topological space and let $\varepsilon > 0$ be arbitrarily small. Clearly in many practical situations we are able to choose $\varepsilon$ so small that replacement of $\varepsilon$ by 0 affects the outcome in a negligible way, i.e., $\varepsilon$ is almost the empty set. However, if we take closures then in $\lambda(\mathcal{T})$, $\bar{\varepsilon} = 1$, i.e., the closure of an almost empty set becomes the entire space!

Finally we remark that in set theory we do not require $\lambda(\mathcal{T})$ and, on the other hand, using $\lambda(\mathcal{T})$ in fuzzy set theory apparently is useless since any operation in $\lambda(\mathcal{T})$ destroys fuzziness by turning fuzzy sets into crisp sets.

**3.12.** *For any topological space $(X, \mathcal{T})$, $\omega(\mathcal{T})$ is the finest structure $\Delta$, both in FTS and in Q-FTS such that $\iota(\Delta) = \mathcal{T}$.*

The result is, by definition of $\omega$ and $\iota$, a triviality. In view of the fact that the functor $\iota$ is inverse to both $\omega$ and $\lambda$ we see that $\omega(\mathcal{T})$ simply is the maximal element in the fiber over $\mathcal{T}$. In contrast $\lambda(\mathcal{T})$ is nothing in particular; being coarser than $\omega(\mathcal{T})$ it is not maximal, and minimal elements do not exist.

**3.13.** *There does not exist a single example of a natural, canonical, explicitly non-fuzzy topological, quasi fuzzy topology, while there exist several examples of canonical fuzzy topologies.*

For the reasons stated in 3.10, the "fuzzy real line" cannot be considered as an example in Q-FTS. In spite of the fact that, also in FTS, much work is of the abstract kind, many natural examples in FTS have been given in [5–8, 20–21, 24, 27, 28]:

1° Every Menger space, $(X, \mathscr{F}, T)$, where $T$ is an Archimedean $t$-norm and $\mathscr{F}$ a statistical metric, generates a canonical probabilistic topology on $X$ [6]. For example, if $\Delta^+$ is the set of all (left continuous) distribution functions $F$ on $\mathbb{R}$ such that $F(0)=0$ and $\mathscr{F}_{T_m}$ is the statistical metric defined in [5] then $(\Delta^+, \mathscr{F}_{T_m}, T_m)$ generates the "positive" fuzzy real line in the category of probabilistic topologies. The topological modification of this structure gives the weak topology on distribution functions.

2° If $(\Omega, \mathscr{A}, P)$ is a probability space and $(X, d)$ a complete separable metric space then put $Z(\Omega, X)$ the set of all $X$-valued, $P$-almost everywhere defined random variables. The mapping $\mathscr{F}$ defined by

$$\mathscr{F}: Z(\Omega,\ X) \times Z(\Omega, X) \to \Delta^+$$

$$\mathscr{F}(f, g)(r) := P\{\omega \mid d(f(\omega), g(\omega)) < r\}$$

is a probabilistic metric. The topological modification of the fuzzy topology generated by this probabilistic metric coïncides with the topology of stochastic convergence [8].

3° If $(X, \mathscr{U})$ is a uniform space and $\Phi(X)$ denotes the set of all upper semi-continuous fuzzy sets on $X$ then in [20, 21] we find two different fuzzy uniformities on $\Phi(X)$, the topological modifications of which in each case coïncide with the Hausdorff–Bourbaki hyperspace structure on the closed subsets of $X \times I$. Moreover, also in each case, the fuzzy structures, when restricted to $\{1_F \mid F \subset X,\ F \text{ closed}\}$, coïncide with the Hausdorff–Bourbaki hyperspace structure on the closed subsets of $X$.

4° If $X$ is any separable metrizable space and $\mathscr{M}(X)$ denotes the collection of all probability measures on $X$ then in [24] we find, on $\mathscr{M}(X)$, a fuzzy topology $\Delta(X)$, which is generated by the subbasis

$$\Sigma(X) := \{\delta_G \mid G \subset X, G \text{ open}\},$$

where $\delta_G(P) := P(G)$ for all $P \in \mathscr{M}(X)$. The topological modification of $\Delta(X)$ coïncides with the weak topology.

5° Let $X$ be any linearly ordered topological space and $\mathscr{L}(X)$ denote the collection of all Radon probability measures on $X$. Let $E(X) := \{(x, y) \mid y$ and $x$ are consecutive$\} \cup \{(x, x) \mid x \in X\}$ and define

$$\rho(P, Q) := \sup_{(x,y) \in E(X)} P((\leftarrow, x[) \wedge Q(]y, \rightarrow)).$$

Then $\rho$ defines an extension of the strict order on $X$ and the right and left sections of this "fuzzy order" generate a fuzzy topology on $\mathscr{X}(X)$. In case $X = \mathbb{R}$ we obtain the fuzzy real line (in FTS!) as in 3.10. Again the topological modification of this structure coïncides with the weak topology [28].

6° If $(X, d)$ is any metric space, $\varepsilon > 0$ is fixed and we put

$$\Delta(\varepsilon) := \{\mu \in I^X \mid \mu \text{ is } 1/\varepsilon\text{-Lipschitz}\},$$

then $\Delta(\varepsilon)$ is a probabilistically $T_m$-uniformizable fuzzy topology on $X$, all of its level topologies coïnciding with the metric topology and which appears to have applications in approximation theory [27].

All these spaces, by their defining mechanism, are in FTS. Only the example in 4° could be adapted to be in Q-FTS\FTS.

## 4. A Question

*Do we need Q-FTS\FTS?*

The examples given in 3.13 give us reasons to believe that FTS is a mathematically worthwhile category to study. Numerous results, by applying the abstract theory of fuzzy topology to these examples, have indeed demonstrated the richness of fuzzy topological concepts when compared to topological concepts. Moreover, the examples show applicability of these concepts to various types of spaces, e.g., statistical metric spaces, spaces of random variables, spaces of probability measures, function spaces, ordered spaces, and metric spaces. These spaces appear in a wide variety of mathematical fields such as statistics, probability theory, analysis, topology, and approximation theory. The same cannot be claimed for Q-FTS\FTS.

Let us illustrate the precise meaning of our question by considering once again the now-classical example of the "fuzzy real line." The basic elements of the "fuzzy real line" are its underlying set $\mathscr{M}(\mathbb{R})$ and a specific subbase which we shall not elaborate on here. Saturating the subbase to obtain a structure of a type in Q-FTS, in FTS, in the category of all probabilistic topological spaces, or in the category of all topologically generated spaces (i.e., essentially TOP itself) one obtains different models of the "fuzzy real line." The finest one simply is $\mathscr{M}(\mathbb{R})$ equipped with the fuzzy topology generated by the weak topology. The other models are equipped with decreasingly coarser fuzzy topologies. In [5–8, 24, 25, 27, 28] the advantage of saturating the given subbase in FTS rather than "essentially" in TOP has been amply demonstrated by means of results on convergence, compactness, and separation. However, not a single result seems to be known providing an argument to go one step further to Q-FTS. None of the afore-mentioned results becomes richer in Q-FTS, on the contrary, as demonstrated in 3.8, convergence becomes pathological. Moreover, this does not seem to be counterbalanced by more positive results of another nature.

Our question therefore precisely is whether there exists an example of a space in Q-FTS\FTS which has certain fundamental good mathematical properties essentially because it is in Q-FTS\FTS, which would loose these properties when saturated to be in FTS?

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